Problem 1

Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \{[f(t)]^2 + [f'(t)]^2\}\, dt \quad \text{for all real } x$$

Solution

In order to remove the integral, we have to differentiate both sides with respect to x and make use of the fundamental theorem of calculus, namely that

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x).$$

We get

$$\begin{aligned} \frac{d}{dx}\left\{[f(x)]^2\right\} &= \frac{d}{dx}\left\{100 + \int_0^x \{[f(t)]^2 + [f'(t)]^2\}\, dt\right\} \\ 2f(x)f'(x) &= 0 + [f(x)]^2 + [f'(x)]^2. \end{aligned}$$

Move all terms to one side.

$$[f'(x)]^2 - 2f(x)f'(x) + [f(x)]^2 = 0$$

Factor.

$$[f'(x) - f(x)]^2 = 0$$

So then

$$\begin{aligned} f'(x) - f(x) &= 0 \\ f'(x) &= f(x) \\ \frac{df}{dx} &= f. \end{aligned}$$

Separate variables.

$$\begin{aligned} \frac{df}{f} &= dx \\ \ln|f| &= x + C \end{aligned}$$

Exponentiate both sides.

$$\begin{aligned} |f| &= e^{x+C} \\ f(x) &= Ae^x \end{aligned}$$

To determine the constant of integration, note that if we plug in $x = 0$ in the starting equation, the whole integral disappears, leaving $[f(0)]^2 = 100$. This means that $A^2 = 100$, or $A = \pm 10$. Therefore,

$$f(x) = \pm 10e^x.$$